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Linear Algebra and its Applications 364 (2003) 223–233

LINEAR ALGEBRA AND ITS APPLICATIONS

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Permanents of woven matrices ☆

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Received 18 January 2001; accepted 22 August 2002

Submitted by R.A. Brualdi

Abstract

A woven matrix, W, is a type of block matrix constructed from an m by n (0, 1)-matrix D with row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$, r_i by r_i matrices R_i ($i = 1, 2, \ldots, m$), and c_j by c_j matrices, C_j ($j = 1, 2, \ldots, n$). Several properties of the determinant and the spectrum of woven matrices are known. In particular, the determinant of a woven matrix is $\pm(\prod_{i=1}^{m} \det R_i)(\prod_{j=1}^{n} \det C_j)$. In this paper it is shown that in general the permanent of W is not determined by the permanents of the R_i and C_j. However, there are instances when

$$\operatorname{per} W = \pm\left(\prod_{i=1}^{m} \operatorname{per} R_i\right)\left(\prod_{j=1}^{n} \operatorname{per} C_j\right). \tag{I}$$

For example, it is shown that (I) holds if at least $m-1$ of the R_i are diagonal matrices. The main result of the paper is a characterization of the D's for which each woven matrix, W, using D satisfies (I). As an application, we determine families of matrices whose permanents can be efficiently computed using determinants.

AMS classification: 15A15

Keywords: Permanent; Woven matrix

☆ This work was supported by Korea Research Foundation Grant (KRF-99-015-DI0003).

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0024-3795/03/$ - see front matter
doi:10.1016/S0024-3795(02)00566-9

1. Introduction

We begin by describing a method, called *weaving*, of constructing new matrices from old ones. The method was originally conceived in order to resolve the existence of certain weighing matrices and related orthogonal matrices [3]. Let $D = [d_{ij}]$ be an m by $n(0, 1)$-matrix with row sums $r_1, r_2, \dots, r_m$ and column sums $c_1, c_2, \dots, c_n$. For $i = 1, 2, \dots, m$, let

$$R_i = \begin{bmatrix} u_{i1} & u_{i2} & \cdots & u_{ir_i} \end{bmatrix}$$

be an r_i by r_i matrix with sth column vector u_{is} and for $j = 1, 2, \dots, n$, let

$$C_j = \begin{bmatrix} v_{1j}^{\mathrm{T}} \\ v_{2j}^{\mathrm{T}} \\ \vdots \\ v_{c_j j}^{\mathrm{T}} \end{bmatrix}$$

be a c_j by c_j matrix with tth row vector v_{tj}^{T}.

For integers i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ define $s(i, j)$ to be the cardinality of the set $\{\ell\colon 1 \leqslant \ell \leqslant j \text{ and } d_{i\ell} = 1\}$.

Similarly, $t(i, j)$ is the cardinality of the set $\{\ell\colon 1 \leqslant \ell \leqslant i \text{ and } d_{\ell j} = 1\}$. The *weaving product of the R_i's and the C_j's by D* is denoted by

$$W(D) = (R_1, \dots, R_m) \otimes_D (C_1, \dots, C_n)$$

and is the m by n block matrix $W(D) = [W_{ij}]$ where

$$W_{ij} = \begin{cases} u_{i,s(i,j)} v_{t(i,j),j}^{\mathrm{T}}, & \text{if } d_{ij} = 1, \\ O, & \text{otherwise.} \end{cases} \tag{1}$$

We call D the *lattice*, R_i the ith *warp*, and C_j the jth *woof* of the weaving. A matrix, $W(D)$, obtained this way is called a *woven matrix*. Note that clearly $W(D)$ is an N by N matrix, where $N = r_1 + r_2 + \cdots + r_m = c_1 + c_2 + \cdots + c_n$.

As an example of a woven matrix, let D be the k by ℓ matrix of all 1's, take all R_i to be a fixed k by k matrix A, and all C_j's to be a fixed ℓ by ℓ matrix B. Then, up to permutation of rows and columns, $W(D)$ is the tensor product of A and B.

The *permutation matrix*, P_D, of D is the m by n block matrix whose (i, j)-block is 0 if $d_{ij} = 0$, and otherwise is the r_i by c_j elementary matrix $E_{s(i,j),t(i,j)}$ which has a 1 in the $(s(i, j), t(i, j))$ position and 0's elsewhere. Note that P_D is an N by N permutation matrix. As observed in [3],

$$W(D) = (R_1 \oplus \cdots \oplus R_m) P_D (C_1 \oplus \cdots \oplus C_n), \tag{2}$$

where $\oplus$ denotes the direct sum. From (2) we see that

$$\det W(D) = (\det P_D)\left(\prod_{i=1}^{m} \det R_i\right)\left(\prod_{j=1}^{n} \det C_j\right)$$
$$= \pm\left(\prod_{i=1}^{m} \det R_i\right)\left(\prod_{j=1}^{n} \det C_j\right). \tag{3}$$

Let $A = [a_{ij}]$ be an n by n matrix. The *permanent* of A, per A, is defined by

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

where σ runs over all permutations of $\{1, \dots, n\}$. Thus, the permanent is like the determinant except it ignores the sign of the permutation σ. Not surprisingly, the determinant and the permanent share many properties. Both are multilinear functions (of the columns) and both are invariant under taking the transpose. However, the permanent fails to inherit the multiplicative property. That is, for square matrices A and B, per AB need not equal (per A)(per B). For this reason, evaluating permanents is considerably more difficult than evaluating determinants. Indeed, permanent evaluation of (0, 1)-matrices is known to be a *#P-complete problem* [5].

In this paper we try to understand the extent to which a permanental analog of (3) holds. The following example shows that, in general, the permanent of a woven matrix is not determined from the permanents of its warps and woofs. Throughout J_k denotes the k by k matrix of all 1's, and $\mathbf{1}_k$ denotes the k by 1 vector of all 1's. Let $D = J_2$, and each of R_1, R_2, C_1 and C_2 be J_2. Then

$$\operatorname{per}((R_1, R_2) \otimes_D (C_1, C_2)) = \operatorname{per} J_4 = 24.$$

Now let

$$R_1' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$\operatorname{per}((R_1', R_2) \otimes_D (C_1, C_2)) = \operatorname{per}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 16.$$

Since per $(R_1') =$ per (R), we conclude that the permanent of a woven matrix $W(D)$ is not determined from the permanents of its warps and woofs.

While the permanental analog of (3) does not hold in general, the following example illustrates that for certain D, (3) holds. Let

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad R_2 = [e]$$

and

$$C_1 = [u], \quad C_2 = \begin{bmatrix} v & w \\ x & y \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{per}((R_1, R_2) \otimes_D (C_1, C_2)) &= \operatorname{per} \begin{bmatrix} ua & bv & bw \\ uc & dv & dw \\ 0 & ex & ey \end{bmatrix} \\ &= (ad + bc)eu(vy + wx) \\ &= \operatorname{per} R_1 \cdot \operatorname{per} R_2 \cdot \operatorname{per} C_1 \cdot \operatorname{per} C_2 \end{aligned}$$

for all $a, b, c, d, e, u, v, w, x, y$.

Define D to be a *multiplicative lattice* if for all warps $R_1, \ldots, R_m$, and all woofs $C_1, \ldots, C_n$

$$\operatorname{per}((R_1, \ldots, R_m) \otimes_D (C_1, \ldots, C_n)) = \left(\prod_{i=1}^{m} \operatorname{per} R_i\right)\left(\prod_{j=1}^{n} \operatorname{per} C_j\right). \tag{4}$$

In Section 2, we characterize the multiplicative lattices D. In Section 3, we use the characterization to give several families of matrices whose permanents are easy to calculate. In particular, we show that if at least $m-1$ of the R_i are diagonal matrices then (4) holds for any woofs $C_1, \ldots, C_n$. In addition, we show how, in special cases, it is possible to convert the problem of computing the permanent of a woven matrix into the problem of computing the determinant of a related woven matrix.

2. Multiplicative lattices

Let $D = [d_{ij}]$ be an m by n matrix. The *bipartite graph* associated with D has vertices $1, 2, \ldots, m,\ 1', 2', \ldots, n'$ and an edge joining (i, j') if and only if $d_{ij} \neq 0$. We say that D is *connected* if and only if its bipartite graph is connected. Thus, D is not connected if and only if there exist permutation matrices P and Q such that PDQ has the form

$$\begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix}.$$

Either D_1 or D_2 can be vacuous by virtue of having no rows or no columns, but neither D_1 nor D_2 is 0 by 0.

Assume further that D is a (0, 1)-matrix. Clearly, if D is not connected then each woven matrix that uses D as its lattice is not connected, and if D is connected and each warp and woof are connected, then the woven matrix $W(D)$ is connected.

In this section, we prove that D is a multiplicative lattice if and only if the bipartite graph associated with D is a forest (that is, a graph with no cycles).

A *diagonal* of the n by n matrix $A = [a_{ij}]$ is a collection of n entries of A, no two of which are in the same row and column. A diagonal is *nonzero* if each of its entries is nonzero. We shall make use of the following well-known result which follows from König's theorem (see Theorem 1.2.1 of [1]).

Proposition 2.1. *Let A be an n by n matrix. Then A has no nonzero diagonal if and only if A has an r by s zero submatrix for some positive integers r and s with $r + s = n + 1$.*

Theorem 2.2. *Let D be an m by n $(0, 1)$-matrix. Then D is a multiplicative lattice if and only if the bipartite graph associated with D is a forest.*

Proof. Without loss of generality we may assume that D is connected. Let the row and column sums of D be $r_1, r_2, \ldots, r_m$ and $c_1, c_2, \ldots, c_n$, respectively. Let $R = (R_1, R_2, \ldots, R_m)$ be a sequence of warps and $C = (C_1, C_2, \ldots, C_n)$ a sequence of woofs for D. Let $\mathscr{E}$ denote the collection of all $E = (E_1, \ldots, E_n)$ such that E_j $(j = 1, \ldots, n)$ is a c_j by c_j $(0, 1)$-matrix with exactly one nonzero entry in each column. Also, let $\mathscr{E}^*$ be the subset of $\mathscr{E}$ consisting of all $E = (E_1, \ldots, E_n)$ where each E_j $(j = 1, \ldots, n)$ is a c_j by c_j permutation matrix. For $E \in \mathscr{E}$ let $E \circ C = (E_1 \circ C_1, E_2 \circ C_2, \ldots, E_n \circ C_n)$ where $\circ$ denotes the Hadamard product of matrices.

Since the permanent is a multilinear function of the columns,

$$\operatorname{per} W(D) = \sum_{E \in \mathscr{E}} \operatorname{per}(R \otimes_D (E \circ C)). \tag{5}$$

Note that if $E \in \mathscr{E}^*$, then

$$\operatorname{per}(R \otimes_D (E \circ C)) = \left(\prod_{i=1}^{m} \operatorname{per} R_i \right) \cdot \prod_{j=1}^{n} \prod_{k=1}^{c_j} c^j_{k,\sigma_j(k)},$$

where the (k, ℓ)-entry of C_j is $c^j_{k,\ell}$.

Hence

$$\sum_{E \in \mathscr{E}^*} \operatorname{per}(R \otimes_D (E \circ C)) = \left(\prod_{i=1}^{m} \operatorname{per} R_i \right) \left(\prod_{j=1}^{n} \operatorname{per} C_j \right). \tag{6}$$

Suppose D is a multiplicative lattice. Set each R_i and C_j to be all 1's matrices. Note that $(\prod_{i=1}^{m} \operatorname{per} R_i)(\prod_{j=1}^{n} \operatorname{per} C_j)$ counts the number of nonzero diagonals of $W(D)$ that have at most one nonzero entry in each block of $W(D)$. Thus, since $W(D)$ is a nonnegative matrix and both (5) and (6) hold, no two 1's of $W(D)$ from the same block lie on a common nonzero diagonal of $W(D)$.

We claim this implies that the bipartite graph associated with D is a tree. It suffices to show that removing any edge of the bipartite graph of D disconnects the graph.

Consider an entry d_{ij} with $d_{ij} = 1$. If $r_i = 1$ or $c_j = 1$ then either vertex i or vertex j' has degree 1, and removing d_{ij} disconnects the bipartite graph of D.

Next assume that $r_i > 1$ and $c_j > 1$. Then the (i, j)-block of $W(D)$ has a 2 by 2 submatrix of all 1's. Since no two 1's of this block lie on a nonzero diagonal of $W(D)$, Proposition 2.1 implies that the complementary submatrix of $W(D)$ contains an r by s zero submatrix Z with $r + s = N - 1$ for some positive integers r and s. Since each block of $W(D)$ is either the zero or all ones matrix, we may assume that Z is composed of blocks of $W(D)$. Hence, there exists a zero submatrix $D[\alpha, \beta]$ of D such that $(i, j) \notin \alpha \times \beta$, and

$$\sum_{k\in\alpha} r_k + \sum_{\ell\in\beta} c_\ell = N - 1. \tag{7}$$

Thus, up to permutation of rows and columns, D has the form

$$\begin{bmatrix} D_1 & D_2 \\ O & D_3 \end{bmatrix}$$

where O is the zero submatrix $D[\alpha, \beta]$. The number of 1's in D_1 is $\sum_{\ell\in\beta} c_\ell$, and the number of 1's in $[D_1\ D_2]$ is

$$\sum_{i\notin\alpha} r_i = N - \sum_{i\in\alpha} r_i.$$

By (7), we conclude that D_2 has exactly one 1. Since this 1 corresponds to the (i, j)-entry of D, removing the edge (i, j') will disconnect the bipartite graph associated with D. Therefore, the bipartite graph associated with D is a tree.

Conversely, assume the bipartite graph associated with D is a tree. Suppose that $d_{ij} = 1$. Removing the edge joining i and j' results in two connnected components, one that contains i and another that contains j'. Let $\alpha \cup \beta$ be the vertices of the component that contains i, where $\alpha \subseteq \{1, 2, \dots, m\}$ and $\beta \subseteq \{1', 2', \dots, n'\}$. Thus, $D[\alpha, \overline{\beta}]$ has exactly one nonzero entry, namely the entry corresponding to d_{ij}. It follows that $\sum_{k\in\alpha} r_k = 1 + \sum_{\ell\in\beta} c_\ell$, and the submatrix, Z, of $W(D)$ corresponding to the row blocks not in α and column blocks in β is a zero submatrix with $\sum_{k\notin\alpha} r_k$ rows and $\sum_{\ell\in\beta} c_\ell$ columns. Thus, the sum of the dimensions of Z is $N - 1$. Proposition 2.1 now implies that no two 1's of the (i, j)-block of $W(D)$ are contained in a nonzero diagonal of $W(D)$. Hence we conclude that each nonzero diagonal of $W(D)$ contains at most one nonzero entry from each block of $W(D)$. Hence for each choice of warps and woofs,

$$\operatorname{per} W(D) = \sum_{E\in\mathscr{E}^*} \operatorname{per}(R \otimes_D (E \circ C)) = \left(\prod_{i=1}^{m} \operatorname{per} R_i\right)\left(\prod_{j=1}^{n} \operatorname{per} C_j\right).$$

Therefore, D is a multiplicative lattice. $\square$

3. Some consequences

Corollary 3.1. *Let P_D be the permutation matrix of a connected lattice D. If P_D is the identity matrix then D is a multiplicative lattice.*

Proof. Assume that $P_D = I_N$. The matrix P_D is an m by n block matrix whose (i, j) block is r_i by c_j. By construction each block of P_D has at most one nonzero entry. This places severe restrictions on D. For example, either $r_1 = 1$ or $c_1 = 1$. Without loss of generality, assume that $r_1 = 1$. Then $r_2 = r_3 = \cdots = r_{c_1-1} = 1$. Hence

$$D = \begin{bmatrix} \mathbf{1}_{c_1-1} & O \\ 1 & d_1^{\mathrm{T}} \\ O & D_2 \end{bmatrix},$$

and I_{N-c_1} is the permutation matrix of

$$D' = \begin{bmatrix} d_1^{\mathrm{T}} \\ D_2 \end{bmatrix}.$$

Thus, the bipartite graph of D is obtained from that of D' by attaching pendant edges to a vertex of D'. It now follows, by an inductive argument, that the bipartite graph of D is a tree. Hence by Theorem 2.2, D is a multiplicative lattice. □

For example, if

$$D = \begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & 1 & & & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & & & 1 & 1 & \end{bmatrix},$$

then $P_D = I_{10}$ and D is a multiplicative lattice.

Now let r and s be positive integers, and let D be the $(r + 1)$ by $(s + 1)$ matrix whose first column and last row have all ones, and all other entries of D are zeros. Since the bipartite graph of D is a tree, D is a multiplicative lattice. Let $R_i = [1]$ for $i = 1, 2, \ldots, r$, $C_j = [1]$ for $j = 2, 3, \ldots, s + 1$, and

$$R_{r+1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{s+1} \end{bmatrix}, \quad C_1 = \begin{bmatrix} u_1^{\mathrm{T}} \\ u_2^{\mathrm{T}} \\ \vdots \\ u_{r+1}^{\mathrm{T}} \end{bmatrix}.$$

Then $W(D) = (R_1, R_2, \ldots, R_{r+1}) \otimes_D (C_1, C_2, \ldots, C_{s+1})$ is the $(r + s + 1)$ by $(r + s + 1)$ matrix such that

$$W(D) = \begin{bmatrix} U' & O \\ v_1 u_{r+1}^{\mathrm{T}} & V' \end{bmatrix},$$

where U' is the matrix obtained from C_1 by deleting its last row, and V' is the matrix obtained from R_{r+1} by deleting its first column. Thus, we have the following:

Proposition 3.2. *Let U be an $(r+1)$ by $(r+1)$ matrix, V an $(s+1)$ by $(s+1)$ matrix, and let*

$$A = \begin{bmatrix} U' & O \\ xy^{\mathrm{T}} & V' \end{bmatrix},$$

where U' is the matrix obtained from U by deleting its last row, and V' is the matrix obtained from V by deleting its first column. Then

$$\operatorname{per} A = (\operatorname{per} U)(\operatorname{per} V).$$

We note that Proposition 3.2 can also be proven directly by noting that each nonzero diagonal of A contains exactly one nonzero entry in the lower left block of A.

More generally, if $k_1, k_2, \ldots, k_t$ are positive integers and

$$D = \begin{bmatrix} \mathbf{1}_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{k_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \mathbf{1}_{k_t} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then D is a multiplicative lattice, and hence we have the following:

Corollary 3.3. *Let $V = [v_1\ v_2\ \cdots v_t]$ be a t by t matrix and for $i = 1, 2, \ldots, t$, let*

$$U_i = \begin{bmatrix} U_i' \\ u_i^{\mathrm{T}} \end{bmatrix}$$

be a k_i by k_i matrix. Then

$$\operatorname{per} \begin{bmatrix} U_1' & O & \cdots & O \\ O & U_2' & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & U_t' \\ v_1 u_1^{\mathrm{T}} & v_2 u_2^{\mathrm{T}} & \cdots & v_t u_t^{\mathrm{T}} \end{bmatrix} = \left(\prod_{i=1}^{t} \operatorname{per} U_i \right) (\operatorname{per} V).$$

We now use Corollary 3.3 to give other instances in which the permanent of a woven matrix is the product of the permanents of its warps and woofs.

Corollary 3.4. *Let D be an $m \times n$ lattice with row sums $r_1,\ r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$, and let R_i and C_j be* ith *warp and* jth *woof of D respectively. If $m-1$ of the R_i's are diagonal matrices, then*

$$\operatorname{per} W(D) = \left(\prod_{i=1}^{m} \operatorname{per} R_i\right)\left(\prod_{j=1}^{n} \operatorname{per} C_j\right).$$

Proof. Since the permanent of a matrix is invariant under row permutations, we may assume that R_i is a diagonal matrix for $i = 1, \ldots, m-1$. Elementary properties of the permanent imply that

$$\begin{aligned}\operatorname{per} W(D) &= \operatorname{per}[(R_1, \ldots, R_m) \otimes_D (C_1, \ldots, C_n)]\\ &= (\operatorname{per} R_1 \cdots \operatorname{per} R_{m-1}) \cdot \operatorname{per}[(I_{r_1}, \ldots, I_{r_{m-1}}, R_m)\\ &\qquad \times \otimes_W (C_1, \ldots, C_n)],\end{aligned}$$

where I_{r_i} is the $r_i \times r_i$ identity matrix. Let $k = r_m$ and assume that the nonzero entries in the mth row of D lie in positions $j_1 < j_2 < \cdots < j_k$. Then it can be verified that the matrix $(I_{r_1}, \ldots, I_{r_{m-1}}, R_m) \otimes_D (C_1, \ldots, C_n)$ is permutation equivalent to

$$\begin{bmatrix} C'_{j_1} & O & O \\ O & \ddots & O \\ O & O & C'_{j_k} \\ v_1 u_{j_1}^{\mathrm{T}} & \cdots & v_k u_{j_k}^{\mathrm{T}} \end{bmatrix} \oplus C_{j_{k+1}} \oplus \cdots \oplus C_{j_n},$$

where

$$C_{j_h} = \begin{bmatrix} C'_{j_h} \\ u_{j_h}^{\mathrm{T}} \end{bmatrix}$$

for each $h = 1, \ldots, k$, and $R_m = [v_1\ \cdots\ v_k]$. Thus from Corollary 3.3, we obtain

$$\begin{aligned}&\operatorname{per}[(I_{r_1}, \ldots, I_{r_{m-1}}, R_m) \otimes_D (C_1, \ldots, C_n)]\\ &\quad = \prod_{i=j_1}^{j_k} \operatorname{per} C_i \cdot (\operatorname{per} R_m) \cdot \prod_{i=j_{k+1}}^{j_n} \operatorname{per} C_i.\end{aligned}$$

Since j_k is taken from $\{1, \ldots, n\}$ for each $k = 1, \ldots, n$, and $j_h \neq j_k$ if and only if $h \neq k$, we have

$$\operatorname{per} W(D) = \left(\prod_{i=1}^{m} \operatorname{per} R_i\right)\left(\prod_{j=1}^{n} \operatorname{per} C_j\right),$$

which completes the proof. □

As the determinant is a multiplicative function, it is relatively simple to calculate the determinant of a matrix. However, the problem of computing the permanent of a square matrix is known to be #P-complete. It has been shown that for some families of matrices it is possible to convert the problem of computing a permanent into the problem of computing a determinant.

More precisely, let M_n denote the set of all n by n real matrices. For a (0,1)-matrix A of order n, we define

$$M_n(A) = \{A \circ X \mid X \in M_n\}.$$

Let H be an $n \times n$ $(0, 1, -1)$-matrix with the property that an entry of H equals 0 if and only if the corresponding entry of A equals 0. Thus H is obtained from A by affixing minus signs to some of its entries. The matrix H *converts the permanent of matrices in* $M_n(A)$ *into the determinant* provided

$$\operatorname{per} X = \det(H \circ X)$$

for all matrices X in $M_n(A)$. It is well known (see [2]) that H converts the permanent of matrices in $M_n(A)$ into the determinant if and only if

$$\operatorname{per} A = |\det H|.$$

If such an H exists, we say that A is *convertible*, and that H is a *conversion* of A. The following result, whose proof follows from definitions and the preceding well known result, shows that each woven matrix whose lattice is multiplicative, and each of whose warps and woofs is convertible is also convertible.

Proposition 3.5. *Let D be an $m \times n$ multiplicative lattice with* ith *row sum r_i and* jth *column sum c_j. Let $R_1, \dots, R_m$ and $C_1, \dots, C_n$ be* (0, 1) *warps and woofs of D each of which is convertible, and let $\widetilde{R}_1, \dots, \widetilde{R}_m, \widetilde{C}_1, \dots, \widetilde{C}_n$ be respective conversions. Then*

$$\left(\widetilde{R}_1, \dots, \widetilde{R}_m\right) \otimes_D \left(\widetilde{C}_1, \dots, \widetilde{C}_n\right)$$

is a conversion of

$$(R_1, \dots, R_m) \otimes_D (C_1, \dots, C_n).$$

For example, let

$$D = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad R_2 = C_1 = C_4 = [1], \quad R_3 = C_2 = C_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and let

$$\widetilde{R}_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \widetilde{R}_2 = \widetilde{C}_1 = \widetilde{C}_4 = [1], \quad \widetilde{R}_3 = \widetilde{C}_2 = \widetilde{C}_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Let

$$W(D) = (R_1, R_2, R_3) \otimes_D (C_1, C_2, C_3, C_4)$$

Then

$$\begin{aligned} \operatorname{per} W(D) &= \left(\prod_{i=1}^{3} \operatorname{per} R_i\right)\left(\prod_{j=1}^{4} \operatorname{per} C_j\right) \\ &= 4 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 1 = 32 \end{aligned}$$

and

$$\begin{aligned} &\det(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3) \otimes_D (\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4) \\ &\quad = \left(\prod_{i=1}^{3} \det \widetilde{R}_i\right)\left(\prod_{j=1}^{4} \det \widetilde{C}_j\right) \\ &\quad = \left(\det \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \det[1] \cdot \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) \\ &\qquad \cdot \left(\det[1] \cdot \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \det[1]\right) \\ &\quad = 4 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 1 = 32. \end{aligned}$$

Hence, $(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3) \otimes_D (\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4)$ is a conversion of $(R_1, R_2, R_3) \otimes_D (C_1, C_2, C_3, C_4)$. We note that not every matrix in $M_n((R_1, R_2, R_3) \otimes_D (C_1, C_2, C_3, C_4))$ is a woven matrix.

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